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# TILINGS WHOSE MEMBERS HAVE FINITELY MANY NEIGHBORS

BY

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ABSTRACT

Let $\mathfrak{C}$ be a tiling of the plane such that each tile of $\mathfrak{C}$ meets at most finitely many other tiles. Then exactly one of the following must occur:

(1) Uncountably many boundary points of $\mathfrak{C}$ belong to no nondegenerate edge of $\mathfrak{C}$, hence $\mathfrak{C}$ has uncountably many singular points; or

(2) Every boundary point of $\mathfrak{C}$ belongs to a nondegenerate edge of $\mathfrak{C}$, moreover, $\mathfrak{C}$ has no singular points.

Furthermore, if $S$ is the set of singular points of $\mathfrak{C}$ and

$$W = \{t : t \in \text{bdry}\, \mathfrak{C} \text{ and } t \text{ belongs to no nondegenerate edge of } \mathfrak{C}\},$$

then $S = \text{cl}\, W$.

## 1. Introduction

We begin with some definitions from [3] and [1]. Family $\mathfrak{C}$ is a *tiling* for the plane if and only if $\mathfrak{C}$ is a collection of closed topological disks having pairwise disjoint interiors for which $\bigcup\{T : T \text{ in } \mathfrak{C}\} = R^2$. Certainly the boundary of a tile is topologically equivalent to a circle, and the intersection of two tiles is a compact, proper subset of their boundaries whose components are simple arcs and singleton point sets. Hence we define an *edge* (nondegenerate edge) of tiling $\mathfrak{C}$ to be a simple arc which is a connected component of the intersection of finitely many tiles of $\mathfrak{C}$, and we define a *vertex* (degenerate edge) of $\mathfrak{C}$ to be a point having this same property. Point $p$ is a *singular point* in tiling $\mathfrak{C}$ if and only if every neighborhood of $p$ meets infinitely many tiles of $\mathfrak{C}$, and a tiling having no singular points is said to be *locally finite*. The reader is referred to [2] for a complete discussion of these ideas and for many illustrative examples.

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Let $W$ denote the set of points in $\{\text{bdry}\, T : T \text{ in } \mathfrak{C}\}$ which belong to no nondegenerate edge of $\mathfrak{C}$, and let $S$ be the set of singular points of $\mathfrak{C}$. In [1], the relationship between $W$ and $S$ was studied, and it was proved that $W$ is countable if and only if $S$ is countable. While in general $W \subseteq S$, the reverse inclusion fails, and in fact an example in [1] shows that $W$ may be empty while $S$ is infinite.

Hence we see that the cardinalities of $W$ and $S$ do not necessarily agree, and a challenging problem is that of characterizing those tilings for which their cardinalities are the same. Our purpose here is to examine one class of tilings having this property. In particular, let $\mathfrak{C}$ be a tiling of the plane, each of whose members has finitely many neighbors. That is, each tile of $\mathfrak{C}$ meets at most finitely many other tiles. Then either both $W$ and $S$ are uncountable or both of them are empty. Moreover, $S = \text{cl}\, W$ in this case.

We will use the following standard terminology throughout the paper: $\text{cl}\, A$ and $\text{bdry}\, A$ denote the closure and boundary of set $A$, respectively. If $\mathfrak{C}$ is a tiling of the plane, we refer to $\bigcup\{\text{bdry}\, T : T \text{ in } \mathfrak{C}\}$ as the boundary of $\mathfrak{C}$, denoted $\text{bdry}\, \mathfrak{C}$.

## 2. The results

The proof of our main theorem will be accomplished by a sequence of preliminary lemmas.

LEMMA 1. *Let $\mathfrak{C}$ be a tiling of the plane such that tile $T$ in $\mathfrak{C}$ meets at most finitely many other tiles. If $x \in \text{bdry}\, T$ and $x$ belongs to no edge (or vertex) of $\mathfrak{C}$, then there is an arc at $x$ in* bdry $T$, *no point of which belongs to any edge (or vertex) of $\mathfrak{C}$.*

PROOF. Observe that since $x$ is in no edge of $\mathfrak{C}$, no other tile can contain $x$. Suppose that no arc exists satisfying the lemma. Then every arc at $x$ in bdry $T$ contains some point belonging to another tile. Hence we may select a sequence $\{x_n\}$ in bdry $T$ converging to $x$ with $x_n \in T \cap T_n$ for an appropriate tile $T_n \neq T$. Since $T$ meets only finitely many tiles, infinitely many of the $T_n$ sets must be equal, so without loss of generality assume that each $T_n$ is tile $T'$. However, then $\{x_n\} \subseteq T \cap T'$ so $x \in T \cap T'$, contradicting our opening observation. Our supposition is false and the lemma is satisfied. Furthermore, it is easy to see that $x$ is relatively interior to such an arc.

It is interesting to observe that Lemma 1 fails in case tile $T$ is allowed to meet infinitely many tiles of $\mathfrak{C}$, as Example 1 illustrates.

EXAMPLE 1. Let $\mathfrak{C}$ be a tiling containing the tiles in Fig. 1. Point $x$ belongs to no edge of $\mathfrak{C}$ yet $x$ is an isolated point having this propery.

LEMMA 2. *Let $\mathfrak{C}$ be a tiling of the plane such that tile $T$ in $\mathfrak{C}$ meets at most finitely many other tiles. If every point of* bdry $T$ *belongs to an edge, then for every $T'$, $T \cap T'$ consists of finitely many components.*

PROOF. Assume on the contrary that $T \cap T'$ consists of infinitely many components, including the countable set of components $\{C_n : n \geqq 1\}$, where $C_{n+1}$ follows $C_n$ relative to an order established on the orientable Jordan curve bdry $T$. Relative to this order, let $D_n$ denote the subset of bdry $T$ which follows $C_n$ and precedes $C_{n+1}$. Then each of $C_n$, $D_n$, $C_{n+1}$ will be an arc. Clearly $D_n \not\subseteq T'$, for otherwise $C_n$ and $C_{n+1}$ would not be distinct components of $T \cap T'$. Furthermore, an infinite subset $S_n$ of $D_n$ is disjoint from $T'$, for otherwise two (closed) components of $T \cap T'$ would share an endpoint, impossible.

For each $n$, select $x_n \in S_n$. Since $x_n$ belongs to an edge, choose tile $T_n \neq T$ with $x_n \in T_n$. (Clearly $T_n \neq T'$.) For $m \neq n$, since our tiles are connected and simply connected, it is not hard to see that $T_m \neq T_n$. However, then we have an infinite family $\{T_n\}$ of tiles which meet $T$, contradicting our hypothesis. We conclude that $T \cap T'$ consists of finitely many components, and the lemma is established.

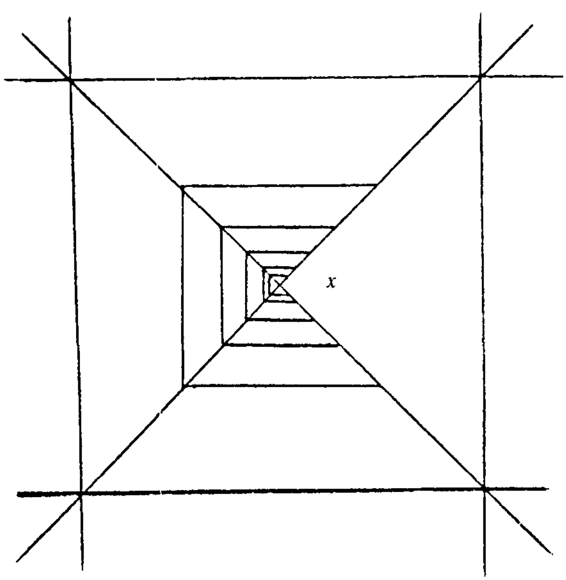


Fig. 1.

LEMMA 3. *Let $\mathfrak{T}$ be a tiling of the plane such that tile $T$ in $\mathfrak{T}$ meets at most finitely many other tiles. If every point of* bdry $T$ *belongs to an edge, then for* $x \in$ bdry $T$, *either $x$ is relatively interior to an edge in* bdry $T$ *or $x$ is an endpoint for* 2 *distinct nondegenerate edges in* bdry $T$.

PROOF. Establish an order on the orientable Jordan curve bdry $T$. For $y$ and $z$ distinct points in bdry $T$, let $\widehat{yz}$ denote the open arc in bdry $T$ from $y$ to $z$, relative to our order. We assume that the lemma fails, to reach a contradiction. Then without loss of generality, we may suppose that for every point $y \in$ bdry $T \sim \{x\}$, $\widehat{yx}$ does not belong to an edge. Choose $x_1 \in$ bdry $T \sim \{x\}$. Since $x_1$ belongs to an edge, we may select tile $T_1 \neq T$ with $x_1 \in T_1$. Furthermore, since $\widehat{x_1x}$ does not belong to any edge, $\widehat{x_1x} \not\subseteq T_1$, and there must be some point $x_2 \in \widehat{x_1x} \sim T_1$. By an obvious induction we obtain sequences $\{x_n\}$ and $\{T_n\}$ with $x_n \in T_n$ and $x_{n+1} \in \widehat{x_nx} \sim T_n$, $n \geqq 1$. Clearly the $x_n$ points are distinct.

Since $T$ meets at most finitely many tiles, infinitely many of the $T_n$ sets must be the same tile, call it $T'$. Moreover, by Lemma 2, $T'$ meets $T$ in finitely many components, so for some $1 \leqq i < j$, $T_i = T_j = T'$, and $x_i$, $x_j$ belong to the same component of $T \cap T'$. By a previous assumption, this edge does not contain $x$, so it does not contain $\widehat{x_jx_i}$, and it must contain $\widehat{x_ix_j}$. However, then

$$x_{j-1} \in \widehat{x_ix_j} \cup \{x_i, x_j\} \subseteq T \cap T' = T \cap T_j,$$

impossible since $x_{j-1} \notin T_j$. We have a contradiction, our assumption must be false, and the lemma is established.

REMARK. It is easy to find examples which show that Lemmas 2 and 3 fail in case $T$ is allowed to meet infinitely many tiles of $\mathfrak{T}$.

LEMMA 4. *Let $\mathfrak{T}$ be a tiling of the plane, and let $N$ be any open disk. Assume that for every tile $T$ in $\mathfrak{T}$, $T \cap N$ meets at most finitely many members of $\mathfrak{T}$, and for each $x \in N \cap$* bdry $\mathfrak{T}$, *$x$ belongs to some edge of $\mathfrak{T}$. Then $N$ contains no singular points.*

PROOF. Minor modifications in Lemmas 1, 2, and 3 above may be used to establish analogous results for points in $N \cap$ bdry $\mathfrak{T}$. To prove Lemma 4, we assume on the contrary that $N$ contains the singular point $x$ of $\mathfrak{T}$, where $x \in$ bdry $T$ for tile $T$. Certainly $x$ cannot be relatively interior to an edge in bdry $T$, so by an adaptation of Lemma 3, $x$ must be an endpoint for 2 nondegenerate edges in bdry $T$. Hence there exist tiles $T_1$ and $T_1'$ such that $x$ belongs to a nondegenerate component of $T \cap T_1$ and similarly $x$ belongs to a nondegenerate component of $T \cap T_1'$. Clearly $T_1 \neq T_1'$. Again by Lemma 3, $x$

must be an endpoint for 2 distinct nondegenerate edges in bdry $T_1$. One of these is the edge contributed by $T \cap T_1$ (already mentioned), and the other is an edge contributed by $T_1 \cap T_2$ for some tile $T_2$. Moreover, since $x$ is a singular point and since our tiles are simply connected, $T_2 \neq T, T'_1$. Repeat for $T'_1$ to obtain $T'_2$. By an obvious induction, we obtain infinitely many tiles $T_n$, $T'_n$ containing $x$, contradicting our hypothesis. Our assumption is false, and we conclude that there are no singular points in $N$.

We are ready to establish the following theorem.

THEOREM 1. *Let $\mathfrak{C}$ be a tiling of the plane such that each tile of $\mathfrak{C}$ meets at most finitely many other tiles. Then exactly one of the following must occur*:

(1) *Uncountably many boundary points of $\mathfrak{C}$ belong to no nondegenerate edge of $\mathfrak{C}$, hence $\mathfrak{C}$ has uncountably many singular points*; *or*

(2) *Every boundary point of $\mathfrak{C}$ belongs to a nondegenerate edge of $\mathfrak{C}$, moreover, $\mathfrak{C}$ has no singular points.*

PROOF. If uncountably many boundary points of $\mathfrak{C}$ belong to no nondegenerate edge, then since each of these points is a singular point ([1, Lemma 1]), the set of singular points of $\mathfrak{C}$ is uncountable, and condition (1) above is satisfied. Otherwise, at most countably many boundary points of $\mathfrak{C}$ belong to no nondegenerate edge. In this case, certainly at most countably many boundary points of $\mathfrak{C}$ belong to no edge (or vertex) of $\mathfrak{C}$, so by Lemma 1, every boundary point of $\mathfrak{C}$ must belong to an edge of $\mathfrak{C}$. However, then Lemma 4 implies that $\mathfrak{C}$ has no singular points, and condition (2) holds.

Moreover, our lemmas may be used to examine the relationship between the set $W$ of boundary points of $\mathfrak{C}$ belonging to no nondegenerate edge of $\mathfrak{C}$ and set $S$ of singular points of $\mathfrak{C}$. For general tilings, $W \subseteq S$ but the reverse inclusion is false, and in fact $W$ may be empty while $S$ is infinite. (See [1, Example 1].) However, in our setting, $\mathrm{cl}\, W = S$.

THEOREM 2. *Let $\mathfrak{C}$ be a tiling of the plane such that each tile of $\mathfrak{C}$ meets at most finitely many other tiles. If $W$ is the set of boundary points of $\mathfrak{C}$ belonging to no nondegenerate edge of $\mathfrak{C}$ and $S$ is the set of singular points of $\mathfrak{C}$, then* $\mathrm{cl}\, W = S$.

PROOF. By [1, Lemma 1], $W \subseteq S$, so $\mathrm{cl}\, W \subseteq S$. To establish the reverse inclusion, let $x$ belong to $R^2 \sim \mathrm{cl}\, W$ to show that $x$ is not a singular point. Select a neighborhood $N$ of $x$ disjoint from $\mathrm{cl}\, W$. Then for every $z \in N \cap \mathrm{bdry}\, \mathfrak{C}$, $z$ belongs to some edge of $\mathfrak{C}$. By Lemma 4, $N$ contains no singular points of $\mathfrak{C}$, so $x \in R^2 \sim S$. Hence $S \subseteq \mathrm{cl}\, W$, and the sets are equal

It is interesting to observe that even in our setting, $S \not\subseteq W$, as Example 2 illustrates.

EXAMPLE 2. Let $\mathscr{C}$ be the collection of copies of the unit square $C \equiv [0,1] \times [0,1]$ arranged in checkerboard fashion. For $n \geqq 3$, let

$$A_n = [1/n, \tfrac{1}{2}] \times [1/n, 1 - 1/n], \qquad A'_n = [\tfrac{1}{2}, 1 - 1/n] \times [1/n, 1 - 1/n],$$

and define tiles $B_n$, $B'_n$ as follows: $B_3 = A_3$, $B'_3 = A'_3$, and for $n \geqq 4$,

$$B_n = \mathrm{cl}(A_n \sim A_{n-1}), \qquad B'_n = \mathrm{cl}(A'_n \sim A'_{n-1}).$$

Finally, define $\mathfrak{T}$ to be the tiling of the plane consisting of tiles in $\mathscr{C} \sim \{C\}$ together with tiles in $\{B_n, B'_n : n \geqq 3\}$. (See Fig. 2.)

Clearly each tile in $\mathfrak{T}$ meets at most finitely many other tiles. However, each vertex $c_i$ of $C$, $1 \leqq i \leqq 4$, is a singular point of $\mathfrak{T}$ which belongs to a nondegenerate edge.

Our final theorem strengthens the result in Theorem 2.

THEOREM 3. *Let $\mathfrak{T}$ be a tiling of the plane such that each tile of $\mathfrak{T}$ meets at most finitely many other tiles. If $U$ is the set of boundary points of $\mathfrak{T}$ belonging to no edge or vertex of $\mathfrak{T}$ and $S$ is the set of singular points of $\mathfrak{T}$, then* $\mathrm{cl}\, U = S$.

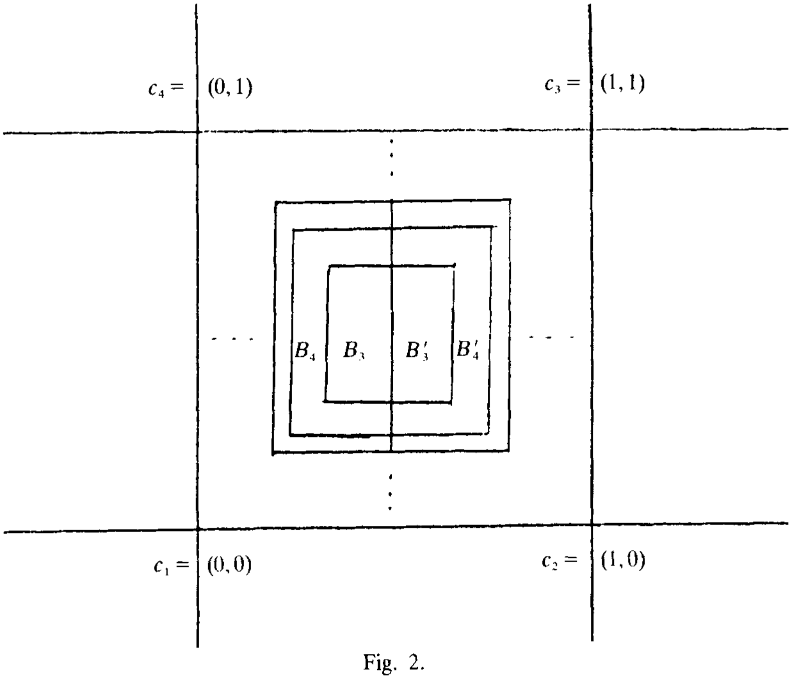


Fig. 2.

PROOF. By remarks in Theorem 2, it suffices to show that for $x$ in $S$, there is a sequence $\{z_n\}$ in $U$ converging to $x$. To begin, observe that for some tile $T$ containing $x$, $x$ is an endpoint for at most one nondegenerate edge in bdry $T$: For if $x$ were an endpoint for two such edges in every tile containing $x$, then an argument like the one in Lemma 4 would yield infinitely many tiles at $x$, clearly impossible. Hence for some tile $T$ and for an arc $A$ at $x$ in bdry $T$, $A$ neither contains nor is contained in a nondegenerate edge at $x$.

In case some subarc of $A$ at $x$ has no point (except possibly $x$) belonging to any edge or vertex, then the lemma is satisfied. Hence we assume that this does not occur. Then there is a sequence $\{x_n\}$ in $A \sim \{x\}$ converging to $x$, with each $x_n$ in some edge or vertex of $\mathfrak{C}$. For future reference, assume that $x_{n+1}$ follows $x_n$ relative to the order established on the Jordan curve bdry $T$, and let $\widehat{x_n x_{n+1}}$ denote the open arc in bdry $T$ from $x_n$ to $x_{n+1}$, relative to this order. Each point $x_n$ belongs to some tile $T_n \neq T$, and since $T$ meets at most finitely many tiles in all, by passing to an appropriate subsequence, we may assume that $T_n = T'$ for every $n$. Of course $x \in T'$, too.

By our choice of $A$, $T \cap T' \cap A$ cannot consist of finitely many components in any neighborhood of $x$: Otherwise, infinitely many $x_n$ points would belong to the same component, and $x$ would lie in this component, giving a nondegenerate edge at $x$ either containing $A$ or contained in $A$, impossible. Thus for every $N$, there is some $n > N$ such that $\widehat{x_n x_{n+1}}$ contains a point $z_n \notin T'$.

Relabeling the $z$ points if necessary, we obtain a sequence $\{z_n\} \subseteq A \sim T'$ with $\{z_n\}$ converging to $x$. Now if infinitely many $z_n$ points should belong to edges (or vertices) of $\mathfrak{C}$, then there would be a subsequence $\{z'_n\}$ of $\{z_n\}$ and a corresponding sequence $\{S_n\}$ in $\mathfrak{C} \sim \{T, T'\}$ with $z'_n \in S_n$. Furthermore, since our tiles are connected and simply connected, the $S_n$ tiles would be distinct, forcing $T$ to meet infinitely many tiles. Since this cannot occur, we conclude that at most finitely many $z_n$ points belong to edges of $\mathfrak{C}$. Hence for $M$ sufficiently large, the sequence $\{z_n : n > M\}$ lies in $U$. By our preliminary remarks, the theorem is established.

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